The Bianchi groups are separable on geometrically finite subgroups.

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Abstract

Let d be a square free positive integer and O_d the ring of integers in $\mathbf{Q}(\sqrt{-d})$. The main result of this paper is to show that the groups $\mathrm{PSL}(2, O_d)$ are subgroup separable on geometrically finite subgroups.

1 Introduction

Let G be a group and H a finitely generated subgroup, G is called *H-subgroup separable* if given any $g \in G \setminus H$, there exists a subgroup $K < G$ of finite index with $H < K$ and $g \notin K$. G is called *subgroup separable* (or *LERF*) if G is H-subgroup separable for all finitely generated $H < G$. Subgroup separability is an extremely powerful property, for instance it is much stronger than residual finiteness. The class of groups for which subgroup separability is known for *all* finitely generated subgroups is extremely small; abelian groups, free groups, surface groups and carefully controlled amalgamations of these, see [12], and [22] for example.

However our motivation comes from 3-manifold topology, where the importance of subgroup separability stems from the well-known fact (cf. [22] and [16]) that it allows passage from immersed incompressible surfaces to embedded incompressible surfaces in finite covers. It therefore makes sense (especially in light of the facts that there are closed 3-manifolds M for which $\pi_1(M)$ is *not* subgroup separable [1]) to ask for separability only for some mildly restricted class of subgroups.

No example of a finite co-volume Kleinian group is known to be subgroup separable. However, in this context the geometrically finite subgroups (especially the geometrically finite surface subgroups) are the intractable and most relevant case in all applications. The reason for this is that the work of Bonahon and Thurston (See [2]) implies that freely indecomposable geometrically infinite subgroups of finite co-volume Kleinian groups are virtual fiber groups, and these are easily seen to be separable. Accordingly, there has been much more attention paid to separating geometrically finite subgroups of finite co-volume Kleinian groups (see [12], [29]). A class of Kleinian groups that have been historically important in the subject are the Bianchi groups. Our main result is the following:

Theorem 1.1 *Let d be a square free positive integer, and O_d the ring of integers in $\mathbf{Q}(\sqrt{-d})$. The Bianchi group $\mathrm{PSL}(2, O_d)$ is H-subgroup separable for all geometrically finite subgroups H.*

A case which has attracted much interest itself is the fundamental group of the figure eight knot complement. This has index 12 in $\mathrm{PSL}(2, O_3)$. Hence we get (see also D. Wise [29] in this case):

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Corollary 1.2 *Let K denote the figure eight knot, then $\pi_1(S^3 \setminus K)$ is H-subgroup separable for all geometrically finite subgroups H.* $\square$

The fundamental group of the Borromean rings is well-known [24] to be a subgroup of index 24 in $\mathrm{PSL}(2, O_1)$, hence we also have:

Corollary 1.3 *The fundamental group of the Borromean rings is subgroup separable on its geometrically finite subgroups.* $\square$

In fact, since it is easy to show that there are infinitely many (2 component) links in S^3 whose complements are arithmetic we deduce,

Corollary 1.4 *There are infinitely many hyperbolic links in S^3 for which the fundamental group of the complement is separable on all geometrically finite subgroups.* $\square$

Our methods also apply to give new examples of cocompact Kleinian groups which are separable on all geometrically finite subgroups.

Theorem 1.5 *There exist infinitely many commensurability classes of cocompact Kleinian groups which satisfy:*

- *they are H-subgroup separable for all geometrically finite subgroups H,*
- *they are not commensurable with a group generated by reflections.*

The second statement of Theorem 1.5 is only included to distinguish the groups constructed from those that Scott's argument ([22] and below) applies to in dimension 3.

In the cocompact setting some interesting groups that we can handle are the following. Let Γ denote the subgroup of index 2 consisting of orientation-preserving isometries in the group generated by reflections in the faces of the tetrahedron in $\mathbf{H}^3$ described as $T_2[2, 2, 3; 2, 5, 3]$ (see [18] for instance for notation). This tetrahedron has a symmetry of order 2, and this symmetry extends to an orientation preserving isometry of the orbifold $Q = \mathbf{H}^3/\Gamma$. Let Γ_0 be the Kleinian group obtained as the orbifold group of this 2-fold quotient. Γ_0 attains the minimal co-volume for an arithmetic Kleinian group (see below), [7]. It is conjectured to attain the smallest co-volume for all Kleinian groups. We show as corollaries of the methods,

Corollary 1.6 Γ_0 *is H-subgroup separable for all geometrically finite subgroups H.*

Corollary 1.7 *Let W denote the Seifert-Weber dodecahedral space, then $\pi_1(W)$ is H-subgroup separable for all geometrically finite subgroups H.*

Although Γ_0 and $\pi_1(W)$ are commensurable with groups generated by reflections, as far as we know they are not commensurable with one where all dihedral angles are $\pi/2$, as is required in applying [22]. Note that by [24] Chapter 13, the group Γ does not split as a free product with amalgamation or HNN-extension since the orbifold $\mathbf{H}^3/\Gamma$ is non-Haken in the language of orbifolds. It is also widely believed that W is non-Haken, and these appear to the first explicit examples of such Kleinian groups separable on geometrically finite subgroups (see also §6.1 for a further example).

An application of Theorem 1.1 that seems worth recording is the following. An obvious subgroup of $\mathrm{PSL}(2, O_d)$ is $\mathrm{PSL}(2, \mathbf{Z})$, and in the context of the Congruence Kernel, Lubotzky [17] asked the following question:

Question: Is the induced map:

$$\eta_d \; : \; \widehat{\mathrm{PSL}}(2,\mathbf{Z}) \to \widehat{\mathrm{PSL}}(2,O_d)$$

injective?

Here $\hat{G}$ denotes the profinite completion (see §8 for details). By a standard reformulation of H-subgroup separable, we are able to give an affirmative answer.

Theorem 1.8 *The map η_d is injective for all d.*

As is pointed out in [17], this has important ramifications for the nature of the Congruence Kernel and the structure of non-congruence subgroups of the Bianchi groups. For example Theorem 1.8 gives another proof that the Bianchi groups do not have the Congruence Subgroup Property [23].

Here is an overview of the paper. The underlying method in proving Theorems 1.1 and 1.5 is to use the arithmetic theory of quadratic forms and their relationship to discrete groups of hyperbolic isometries to inject (up to commensurability) Kleinian groups into a fixed finite co-volume group acting on a higher dimensional hyperbolic space commensurable with a group generated by reflections in an all right polyhedron in $\mathbf{H}^n$. This part of the proof hinges upon Lemmas 4.4 and 4.6.

The ambient reflection group is then shown to be subgroup separable on its geometrically finite subgroups. This has its origins in [22] where it is shown that in dimension 2, cocompact Fuchsian groups are subgroup separable. In §3, we generalise this to all dimensions. The situation is actually a good deal more delicate than is generally appreciated; in particular, in Scott's article [22] various statements are made that have been taken to suggest that his methods extend to higher dimensions and that they could be used to separate geometrically finite subgroups inside Kleinian groups commensurable with groups generated by reflections in all right ideal polyhedra in $\mathbf{H}^3$. However it seems to us (and has been confirmed by Scott) that this is not so.

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2 Preliminaries

We recall some facts about discrete groups of isometries of hyperbolic spaces, and arithmetic hyperbolic n-manifolds, [20] and [3]. A reasonable reference that contains information on both these topics is [28].

2.1

Let f_n be the $(n+1)$-dimensional quadratic form $< 1, 1 \ldots, 1, -1 >$. The orthogonal group of this form is simply $O(n,1;\mathbf{R})$. This has four connected components. Passing to $\mathrm{SO}(n,1;\mathbf{R})$, this has two connected components, and the connected component of the identity in $\mathrm{SO}(n,1;\mathbf{R})$, denoted $\mathrm{SO}_0(f_n;\mathbf{R})$ (which has finite index in $\mathrm{O}(n,1;\mathbf{R})$), may be identified with $\mathrm{Isom}_+(\mathbf{H}^n)$; it preserves the upper sheet of the hyperboloid $f_n(x) = -1$ and the orientation. Given a (discrete) subgroup G of $\mathrm{O}(n,1;\mathbf{R})$, $G \cap \mathrm{SO}_0(n,1;\mathbf{R})$ has finite index in G.

2.2

A Kleinian group will always refer to a discrete group of orientation preserving isometries of $\mathbf{H}^3$. Throughout the paper we often pass between models of $\mathbf{H}^3$, and use the term Kleinian group in both settings. Hopefully no confusion will arise.

A Kleinian group Γ is *geometrically finite* if either of the following equivalent statements hold (see [20]):

1. Γ admits a finite sided Dirichlet polyhedron for its action on $\mathbf{H}^3$.
2. Let $C(\Gamma)$ denote the convex core of $\mathbf{H}^3/\Gamma$, then for all $\epsilon > 0$ the ϵ-neighbourhood, $N_\epsilon(C(\Gamma))$ has finite volume.

In higher dimensions, geometrical finiteness has been more problematical, see [5] for example. For instance, the generalization of 1 above (which is the classical statement in dimensions 2 and 3) becomes more restrictive in higher dimensions, cf. [5]. However, if we insist that Γ is *finitely generated*, then the statement 2 above suffices as a definition for Γ to be geometrically finite, see [5]. Henceforth, the term geometrically finite always refers to this. That finite generation is important follows from work of [13].

We will also make use of the following equivalent statement of geometrical finiteness ($GF2$ of [5]). This requires some terminology from [5].

Suppose that Γ is a discrete subgroup of $\mathrm{Isom}_+(\mathbf{H}^n)$, and $\Lambda(\Gamma)$ is its limit set. Let $p \in \Lambda(\Gamma)$ be a parabolic fixed point and

$$\mathrm{Stab}_p(\Gamma) = \{\gamma \in \Gamma \ : \ \gamma p = p\}.$$

p (as above) is called *bounded* if $(\Lambda(\Gamma) \setminus p)/\mathrm{Stab}_p(\Gamma)$ is compact. Then [5] shows:

Lemma 2.1 *Suppose that Γ is a discrete subgroup of $Isom_+(\mathbf{H}^n)$. Then Γ is geometrically finite if and only if the limit set consists of conical limit points and bounded parabolic fixed points.* $\square$

As is pointed out in [5] the notion of bounded parabolic fixed point can be stated as: p is a bounded parabolic fixed point if and only if given a minimal plane τ which is invariant for the action of $\mathrm{Stab}_\Gamma(p)$ then there is a constant r with the property that $\Lambda(\Gamma) \setminus \{p\}$ lies inside an r-neighbourhood (measured in a Euclidean metric on $\mathbf{R}^{n-1}$) of the plane τ.

2.3

We will require some notions from the theory of quadratic forms, we refer to [14] as a standard reference.

We first fix some notation, if k is a number field, then R_k will denote its ring of integers.

Definition. Two n-dimensional quadratic forms f and q defined over a field k (with associated symmetric matrices F and Q) are *equivalent* over k if there exists $P \in \mathrm{GL}(n, k)$ with $P^t F P = Q$.

Let f be a quadratic form in $n+1$ variables with coefficients in a real number field k, with associated symmetric matrix F, and let

$$\mathrm{SO}(f) = \{X \in \mathrm{SL}(n+1, \mathbf{C}) \mid X^t F X = F\}$$

be the Special Orthogonal group of f. This is an algebraic group defined over k, and $\mathrm{SO}(f; R_k)$ is an arithmetic subgroup of $\mathrm{SO}(f; \mathbf{R})$, [4] or [3]. In particular $\mathrm{SO}(f; \mathbf{R})/\mathrm{SO}(f; R_k)$ has finite volume with respect to a suitable measure.

Lemma 2.2 *In the notation above,* $\mathrm{SO}(f; \mathbf{R})$ *is conjugate to* $\mathrm{SO}(q; \mathbf{R})$*,* $\mathrm{SO}(f; k)$ *is conjugate to* $\mathrm{SO}(q; k)$*, and* $\mathrm{SO}(f; R_k)$ *is conjugate to a subgroup of* $\mathrm{SO}(q; k)$ *commensurable with* $\mathrm{SO}(q; R_k)$*.*

Proof. We do the k-points case first, the real case follows the same line of argument. Thus, let $A \in \mathrm{SO}(q;k)$, then,

$$(PAP^{-1})^t F(PAP^{-1}) = (P^{-1})^t A^t (P^t FP) AP^{-1} = (P^{-1})^t A^t QAP^{-1}.$$

Since $A \in \mathrm{SO}(q;k)$, this gives,

$$(P^{-1})^t QP^{-1} = F$$

by definition of P. Since P has k-rational entries, we have that $PAP^{-1} \in \mathrm{SO}(f;k)$. The reverse inclusion is handled similarly.

To handle the second statement, observe that by considering the denominators of the entries of P and P^{-1}, we can choose a deep enough congruence subgroup Γ in $\mathrm{SO}(q;R_k)$ so that that $P\Gamma P^{-1}$ will be a subgroup of $\mathrm{SO}(f;R_k)$. The index is necessarily finite since both are of finite co-volume. $\square$

Assume now that k is totally real, and let f be a form in $n+1$-variables with coefficients in k, and be equivalent over $\mathbf{R}$ to the form f_n (as in §2.1). Furthermore, if $\sigma : k \to \mathbf{R}$ is a field embedding, then the form f^σ obtained by applying σ to f is defined over the real number field $\sigma(k)$. We insist that for embeddings $\sigma \neq id$, f^σ is equivalent over $\mathbf{R}$ to the form in $(n+1)$-dimensions, of signature $(n+1,0)$. Since f is equivalent over $\mathbf{R}$ to f_n, it from follows Lemma 2.2 that $\mathrm{SO}(f;\mathbf{R})$ is conjugate in $\mathrm{GL}(n+1,\mathbf{R})$ to $\mathrm{SO}(f_n;\mathbf{R})$. From the discussion in §2.1, we deduce from [4] (or [3]) that $\mathrm{SO}_0(f;R_k)$ defines an arithmetic lattice in $\mathrm{Isom}_+(\mathbf{H}^n)$. For n odd, this gives only a subclass of arithmetic lattices in $\mathrm{Isom}_+(\mathbf{H}^n)$ (see [28] pp 221–222).

The group $\mathrm{SO}_0(f;R_k)$ (and hence the conjugate in $\mathrm{Isom}_+(\mathbf{H}^n)$) is cocompact if and only if the form f does not represent 0 non-trivially with values in k, see [4]. Whenever $n \geq 4$, the lattices constructed above are non-cocompact if and only if the form has rational coefficients, since it is well known every quadratic form over $\mathbf{Q}$ in at least 5 variables represents 0 non-trivially, see [14].

We make some comments on arithmetic hyperbolic 3-manifolds constructed in the above manner. It is a consequence of the results in [18] that when $n = 3$, the above class of arithmetic Kleinian groups coincide precisely with those arithmetic Kleinian groups containing a non-elementary Fuchsian subgroup and for which the invariant trace field is quadratic imaginary. As we will require it we state a corollary of the main result of [18]. This ties up the $\mathrm{PSL}(2)$ and $\mathrm{SO}(3,1)$ descriptions of certain arithmetic Kleinian groups we will need.

Theorem 2.3 *Let a, b and c be integers with $a < 0$ and $b, c > 0$. Let q be the quadratic form $< 1, a, b, c >$. Then $\mathrm{SO}_0(q,\mathbf{Z})$ defines an arithmetic Kleinian subgroup of $\mathrm{PSL}(2,\mathbf{C})$ with invariant trace-field $\mathbf{Q}(\sqrt{abc})$ and invariant quaternion algebra with Hilbert Symbol $\left(\frac{-ac,-bc}{\mathbf{Q}(\sqrt{abc})}\right)$.* $\square$

Indeed the correspondence above is a bijective correspondence between commensurability classes in the two models.

3 All right reflection groups and separability

Suppose that P is a compact or ideal polyhedron (ie at least one vertex lies at the sphere-at-infinity) in $\mathbf{H}^n$ all of whose dihedral angles are $\pi/2$. Henceforth we call this an *all right polyhedron.* Then the Poincaré polyhedron theorem implies that the group generated by reflections in the co-dimension one faces of P is discrete and a fundamental domain for its action is the polyhedron P, that is to say, we obtain a tiling of hyperbolic n-space by tiles all isometric to P. Let the group so generated be denoted by $G(P)$.

Theorem 3.1 *The group $G(P)$ is H-subgroup separable for every finitely generated geometrically finite subgroup $H < G(P)$.*

It seems to be a folklore fact that this theorem follows easily from the ideas contained in [22]; however it seems to us that this is not the case and we include a complete proof. One piece of terminology we require is the following. Any horospherical cusp cross-section of a hyperbolic n-orbifold of finite volume is finitely covered by the $n-1$-torus (see [20] Chapter 5). We say a cusp of a hyperbolic n-orbifold is of *full rank* if it contains $\mathbf{Z}^{n-1}$ as a subgroup of finite index. Otherwise the cusp is said not to be of full rank.

Proof of 3.1 The proof breaks up into various cases which we deal with in ascending order of difficulty. All proofs hinge upon the observation (see [22]) that the separability of H is equivalent to the following:

Suppose that we are given a compact subset $X \subset \mathbf{H}^n/H$. Then there is a finite index subgroup $K < G(P)$, with $H < K$ and with the projection map $q : \mathbf{H}^n/H \longrightarrow \mathbf{H}^n/K$ being an embedding on X.

We sum up the common strategy which achieves this. The group H is geometrically finite and one can enlarge its convex hull in $\mathbf{H}^n/H$ so as to include the compact set X in a convex set contained in $\mathbf{H}^n/H$; this convex set lifts to an H-invariant convex set inside $\mathbf{H}^n$. One then defines a coarser convex hull using only the hyperbolic halfspaces bounded by totally geodesic planes which come from the P-tiling of $\mathbf{H}^n$; this hull is denoted by $H_P(C^+)$. This hull is H-invariant and the key point is to show that $H_P(C^+)/H$ only involves a finite number of tiles. It is the mechanics of achieving this that vary depending on the nature of P and H; the remainder of the proof follows [22] and is an elementary argument using the Poincaré polyhedron theorem and some covering space theory. The details are included in the first argument below.

3.1 P is compact.

Let C be a very small neighbourhood of the convex hull of H, regarded as a subset of $\mathbf{H}^n$. In our setting, the group $G(P)$ contains no parabolic elements so that the hypothesis implies that C/H is compact.

The given set X is compact so that there is a t with the property that every point of X lies within a distance t of C/H. Let C^+ be the $10t$ neighbourhood of C in $\mathbf{H}^n$. This is still a convex H-invariant set and C^+/H is a compact convex set containing X.

As discussed above, take the convex hull $H_P(C^+)$ of C^+ in $\mathbf{H}^n$ using the half spaces coming from the P-tiling of $\mathbf{H}^n$. By construction $H_P(C^+)$ is a union of P-tiles, is convex and H-invariant. The crucial claim is:

Claim: $H_P(C^+)/H$ involves only a finite number of such tiles.

To see this we argue as follows.

Fix once and for all a point in the interior of a top dimensional face of the tile and call this its *barycentre*. The tiles we use actually often have a geometric barycentre (i.e. a point which is equidistant from all of the faces) but such special geometric properties are not used; it is just a convenient reference point.

Our initial claim is that if the barycentre of a tile is too far away from C^+, then it cannot lie in $H_P(C^+)$.

The reason for this is the convexity of C^+. If a is a point in $\mathbf{H}^n$ not lying in C^+ then there is a unique point on C^+ which is closest to a. Moreover, if this distance is R, then the set of points

distance precisely R from a is a sphere touching C^+ at a single point p on the frontier of C^+ and the geodesic hyperplane tangent to the sphere at this point is the (generically unique) supporting hyperplane separating C^+ from a.

Suppose then that P^* is a tile whose barycentre is very distant from C^+. Let a^* be the point of P^* which is closest to C^+ and let p be a point on the frontier of C^+ which is closest to P^*. As noted above, there is a geodesic supporting hyperplane $\mathcal{H}_p$ through p which is (generically) tangent to C^+ and separates C^+ from a^*. Let the geodesic joining a^* and p be denoted by γ. Note that since p is the point of C^+ closest to a^*, γ is orthogonal to $\mathcal{H}_p$.

If a^* happens to be in the interior of a tile face of P^*, then this tile face must be at right angles to γ, since a^* was closest. Let $\mathcal{H}_{a^*}$ be the tiling plane defined by this tile face. Since in this case γ is at right angles to both $\mathcal{H}_{a^*}$ and $\mathcal{H}_p$, these planes are disjoint and so the tiling plane separates P^* from C^+ as required.

If a^* is in the interior of some smaller dimensional face, σ, then the codimension one faces of P^* which are incident at σ cannot all make small angles with γ since they make right angles with each other. The hyperplane $\mathcal{H}$ which makes an angle close to $\pi/2$ plays the role of $\mathcal{H}_{a^*}$ in the previous paragraph. The reason is that since a^* and p are very distant and the planes $\mathcal{H}_p$ and $\mathcal{H}$ both make angles with γ which are close to $\pi/2$, the planes are disjoint and we see as above that P^* cannot lie in the tiling hull in this case either.

The proof of the claim now follows, as there can be only finitely many barycentres near to any compact subset of $\mathbf{H}^n/H$.

The proof of subgroup separability now finishes off as in [22]. Let K_1 be the subgroup of $G(P)$ generated by reflections in the sides of $H_P(C^+)$. The Poincaré polyhedron theorem implies that $H_P(C^+)$ is a (noncompact) fundamental domain for the action of the subgroup K_1. Set K to be the subgroup of $G(P)$ generated by K_1 and H, then $\mathbf{H}^n/K = H_P(C^+)/H$ so that K has finite index in $G(P)$. Moreover, the set X embeds as required. $\Box$

3.2 P is an ideal all right polyhedron.

Subcase A: H has no cusps.

This case is very similar to the case that $G(P)$ is cocompact since in the absence of cusps, the core of H is actually compact. We form the set C^+ as above. The set C^+/H is still compact so that it only meets a finite number of tiles and we choose a constant K so that the barycentre of each such tile is within distance K of C^+/H.

Now we repeat the argument above, with the extra care that one should only look at tiles in $\mathbf{H}^n$ whose barycentres are at distance from C^+ much larger than K; this ensures that such a tile cannot meet C^+ and the rest of the argument is now identical.

Subcase B: H has cusps which are all of full rank.

In this case the core C^+/H is no longer compact, but by geometrical finiteness it has finite volume. The thick part of this core is compact and can be covered by a finite number of tiles. Also the thin part can be covered by a finite number of tiles; one sees this by putting the cusp of H at infinity, the cusp has full rank so there is a compact fundamental domain for its action and this fundamental domain meets only a finite number of tiles.

Choose K for this finite collection of tiles, then argue as in Subcase A.

Subcase C: H has a cusp of less than full rank.

The idea in this case is to enlarge H to a group H^* which now only has full rank cusps in such a way

that the compact set X continues to embed in the quotient $\mathbf{H}^n/H^*$; we then argue as in Subcase B. The argument follows a line established in [8]. We assume that H has a single cusp of less than full rank, the case of many cusps is handled by successive applications of this case.

To this end, consider the upper n-space model for $\mathbf{H}^n$ arranged so that ∞ is a parabolic fixed point for H. Denote the limit set of H by $\Lambda(H)$. By Lemma 2.1 and the remarks following it, if τ is a minimal plane which is invariant for the action of $\mathrm{Stab}_H(\infty)$ then there is a constant r with the property that $\Lambda(H) - \infty$ lies inside an r-neighbourhood (measured in a Euclidean metric on $\mathbf{R}^{n-1}$) of the plane τ.

We now sketch the construction of [8]. We have already observed that C^+/H is a finite volume hyperbolic n-orbifold; one can therefore define a "thickness" for this hull, that is to say, there is a constant c_1 so that every point on the upper hypersurface of C^+ is within distance at most c_1 of some point on the lower hypersurface. We may as well suppose that c_1 is fairly large, say at least 10.

Choose a horoball N' in $\mathbf{H}^n/H$ which is so small that it is a very long way from the original compact set X. By shrinking further, we arrange that the distance between any two preimages of N' in $\mathbf{H}^n$ is at least, say $1000c_1$. Now shrink further and find a horoball $N \subset N'$ so that ∂N is distance $1000c_1$ from $\partial N'$. It follows that when we look in the universal covering, if a preimage of N is not actually centred on some point of $\Lambda(H)$, then it is distance at least $750c_1$ from C^+.

Using geometrical finiteness in the form given by Lemma 2.1, we may find a pair of $\mathrm{Stab}_H(\infty)$-invariant, parallel, totally geodesic hyperbolic $n-1$-planes, both passing through ∞ which are distance $10r$ apart in the Euclidean metric on $\mathbf{R}^{n-1}$ and contain $\Lambda(H) - \infty$. By moving the planes further apart if necessary, we may assume that the slab of hyperbolic n-space between them contains all of C^+.

If we denote the dimension of the minimal plane τ by $n-1-k$, then we may choose k such pairs of planes so that the union of these of $2k$ planes cuts out subset of $\mathbf{R}^{n-1}$ which has the form $[0,1]^k \times \mathbf{R}^{n-1-k}$ containing all of $\Lambda(H) - \infty$ and so that the slab of hyperbolic n-space cut out by these planes contains all of C^+. Denote this slab by Σ.

Let N_∞ be the preimage of N centred at ∞; ∂N_∞ is a copy of $\mathbf{R}^{n-1}$ equipped with a Euclidean metric coming from the restriction of the metric coming from $\mathbf{H}^n$. By choosing translations in $\mathrm{Stab}_{G(P)}(\infty)$ which move points a very long distance in the Euclidean metric on ∂N_∞, we can augment $\mathrm{Stab}_H(\infty)$ to form a new subgroup $\mathrm{Stab}_H(\infty)^* \subset \mathrm{Stab}_{G(P)}(\infty)$, now of full rank, with the property that any element of $\mathrm{Stab}_H(\infty)^*$ either stabilises the slab Σ (this is the case that the translation in question in fact lies in the subgroup $\mathrm{Stab}_H(\infty)$) or moves it a very long distance from itself, measured in the Euclidean metric of ∂N_∞.

Define the subgroup H^* to be the group generated in $G(P)$ by H and $\mathrm{Stab}_H(\infty)^*$. It follows exactly as in [8] that the group H^* is geometrically finite and leaves invariant a convex set C^* which is slightly larger than the H^*-orbit of $C^+ \cup N_\infty$; moreover the set X embeds into C^*/H^*. By construction H^* has a full rank cusp. $\square$

3.3

We now discuss the existence of all right polyhedra in hyperbolic spaces. It is well-known that such polyhedra cannot exist for large dimensions.

We first fix some notation. Let $\mathcal{P}$ be a convex polyhedron in $\mathbf{H}^n$ with a finite number of codimension one faces, all of whose dihedral angles are integer submultiples of π. Denote by $G(\mathcal{P})$ (resp. $G^+(P)$) the group generated by reflections in the codimension one faces of $\mathcal{P}$ (resp. subgroup of index 2 consisting of orientation-preserving isometries). $G(P)$ is discrete by Poincaré's Theorem (see [20] Chapter 7 or [28]).

Ideal 24-cell in hyperbolic 4-space:

Let P denote the all right ideal 24 cell P in $\mathbf{H}^4$ (cf. [20], Example 6 p. 273, p. 509). We have the following lemma (see [21]) which records some arithmetic data associated to $G(P)$.

Lemma 3.2 *$G^+(P)$ is an arithmetic lattice in $Isom_+(\mathbf{H}^4)$. It is a subgroup of finite index in* $\mathrm{SO}_0(f_4; \mathbf{Z})$. $\square$

A compact all right 120-cell in hyperbolic 4-space:

Let D denote the regular 120-cell in $\mathbf{H}^4$ with all right dihedral angles, see [10]. This has as faces 3-dimensional all right dodecahedra. D is built from 14400 copies of the $\{4,3,3,5\}$ Coxeter simplex, Σ in $\mathbf{H}^4$. We fix the following notation to be used throughout. $\mathcal{O}$ will denote the ring of integers in $\mathbf{Q}(\sqrt{5})$, and τ will denote the non-trivial Galois automorphism of $\mathbf{Q}(\sqrt{5})$.

Lemma 3.3 *The group $G^+(D)$ is an arithmetic lattice in $Isom_+(\mathbf{H}^4)$. It is commensurable with the group $\mathrm{SO}_0(f; \mathcal{O})$ where f is the 5-dimensional quadratic form $< 1, 1, 1, 1, -\phi >$, and $\phi = \frac{1+\sqrt{5}}{2}$.*

Proof. By Vinberg's criteria [26], the group generated by reflections in the faces of Σ is arithmetic. By the remarks above, $G^+(D)$ is also arithmetic. The description given follows from [6], see also [28] p. 224. $\square$

An all right ideal polyhedron in hyperbolic 6-space:

In $\mathbf{H}^6$ there is a simplex Σ with one ideal vertex given by the following Coxeter diagram (see [20] p. 301).

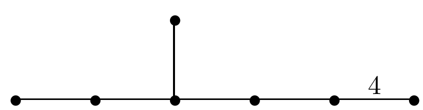

Figure 1

Notice that deleting the right most vertex of this Coxeter symbol gives an irreducible diagram for a finite Coxeter group, namely E_6. This group has order $2^7.3^4.5$.

We will make use of the following.

Lemma 3.4 *(i) $G^+(\Sigma) = \mathrm{SO}_0(f_6; \mathbf{Z})$.*

(ii) There is an all right polyhedron Q built from $2^7.3^4.5$ copies of Σ. In particular the reflection group $G(Q)$ is commensurable with $\mathrm{SO}_0(f_6; \mathbf{Z})$.

Proof. The first part is due to Vinberg [27], and also discussed in [20] p. 301. For the second part, as noted above, if one deletes the face F of the hyperbolic simplex corresponding to the right hand vertex to the given Coxeter diagram, the remaining reflection planes pass through a single (finite) vertex and these reflections generate the finite Coxeter group E_6. Take all the translates of the simplex by this group; this yields a polyhedron whose faces all correspond to copies of F. Two such copies meet at an angle which is twice the angle of the reflection plane of the hyperbolic simplex which lies between them. One sees from the Coxeter diagram that the plane F makes angles $\pi/2$ and $\pi/4$ with the other faces of the hyperbolic simplex, so the resulting polyhedron is all right as required. $\square$

Remark. The polyhedron is finite covolume since there is only one infinite vertex: deleting the plane

corresponding to the left hand vertex of the Coxeter group is the only way of obtaining an infinite group and this group is a 5 dimensional Euclidean Coxeter group. (See Theorem 7.3.1 Condition (2) of [20]) The other Coxeter diagrams of [20] shows that there are ideal all right polyhedra in $\mathbf{H}^k$ at least for $2 \leq k \leq 8$.

4 Proof of Theorem 1.1

The subsections that follow collect the necessary material to be used in the proof.

4.1

We need the the following standard facts.

Lemma 4.1 *Let G be a group and let $H < K < G$. If G is H-subgroup separable then K is H-subgroup separable.*

Proof. Let $k \in K \setminus H$. Since G is H-subgroup separable there is a finite index subgroup $G_0 < G$ with $H < G_0$ but $k \notin G_0$. Then $G_0 \cap K$ is a subgroup of finite index in K separating H from k as required. □

Lemma 4.2 *Let G be a finite co-volume Kleinian group and let K be a subgroup of finite index. If K is H-subgroup separable for all geometrically finite subgroups $H < K$, then G is H-subgroup separable for all geometrically finite subgroups $H < G$.*

Proof. It is a standard fact (see for example Lemma 1.1 of [22]) that if K is subgroup separable and $K \leq G$ with K being of finite index, then G is subgroup separable. The proof of this result applies verbatim to Lemma 4.2 after one notes that the property of being geometrically finite is preserved by super- and sub- groups of finite index. □

Lemma 4.3 *Let Γ be a discrete subgroup of Isom$_+(\mathbf{H}^n)$ of finite co-volume and Γ_0 a geometrically finite subgroup of Γ fixing a totally geodesic copy of hyperbolic 3-space in $\mathbf{H}^3$. Then Γ_0 is geometrically finite as a subgroup of Isom$_+(\mathbf{H}^n)$.*

Proof. Given a geometrically finite hyperbolic (n-1)-orbifold $\mathbf{H}^{n-1}/G$ this can be seen to be geometrically finite as a quotient of $\mathbf{H}^n$ by observing that an ϵ-neighbourhood of the core in $\mathbf{H}^n$ is isometric to (core in $\mathbf{H}^{n-1}$) $\times I$ since G fixes a genuine co-dimension one geodesic sub-hyperbolic space. That the n-dimensional core has finite volume now follows from the fact that the $(n-1)$-dimensional core does. The proof of the statement we require follows from this and induction. Alternatively one could use Lemma 2.1 and observe that the properties of conical limit points and bounded parabolic fixed points will be preserved. □

4.2

The key lemma is the following.

Lemma 4.4 *Let f be the quadratic form $< 1, 1, 1, 1, 1, 1, -1 >$. Then for all d,* $\mathrm{SO}(f; \mathbf{Z})$ *contains a group G_d which is conjugate to a subgroup of finite index in the Bianchi group* $\mathrm{PSL}(2, O_d)$.

The proof requires an additional lemma. Assume that j is a diagonal quaternary quadratic form with integer coefficients of signature $(3, 1)$; so that j is equivalent over $\mathbf{R}$ to the form $< 1, 1, 1, -1 >$. Let $a \in \mathbf{Z}$ be a square-free positive integer and consider the seven dimensional form $j_a =< a, a, a >$

$\oplus\ j$, where $\oplus$ denotes orthogonal sum. Being more precise, if we consider the 7-dimensional $\mathbf{Q}$-vector space V equipped with the form j_a there is a natural 4-dimensional subspace V_0 for which the restriction of the form is j. Using this it easily follows that,

Lemma 4.5 *In the notation above, the group* $\mathrm{SO}(j;\mathbf{Z})$ *is a subgroup of* $\mathrm{SO}(j_a;\mathbf{Z})$. $\square$

Proof of Lemma 4.4

Let p_d be the quaternary form $< d,1,1,-1 >$. Notice that this form represents 0 non-trivially, and hence the corresponding arithmetic group $\mathrm{SO}_0(p_d;\mathbf{Z})$ is non-cocompact. By Theorem 2.3 for example, this implies that $\mathrm{SO}_0(p_d;\mathbf{Z})$ is commensurable with some conjugate of an appropriate image of the Bianchi group $\mathrm{PSL}(2,O_d)$. The key claim is that $q_d =< d,d,d > \oplus\ p_d$ is equivalent over $\mathbf{Q}$ to the form f.

Assuming this claim for the moment, by Lemma 2.2 we deduce that there exists $R_d \in \mathrm{GL}(7,\mathbf{Q})$ such that $R_d\mathrm{SO}(q_d;\mathbf{Z})R_d^{-1}$ and $\mathrm{SO}(f;\mathbf{Z})$ are commensurable. This together with Lemma 4.5 gives the required group G_d.

To prove the claim, since every positive integer can be written as the sum of four squares, write $d = w^2 + x^2 + y^2 + z^2$. Let A_d be the 7×7 matrix

$$\begin{pmatrix} w & x & y & z & 0 & 0 & 0 \\ -x & w & -z & y & 0 & 0 & 0 \\ -y & z & w & -x & 0 & 0 & 0 \\ -z & -y & x & w & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Note A_d has determinant d^2, so is in $\mathrm{GL}(7,\mathbf{Q})$. Let F the diagonal matrix associated to the form f and Q_d be the 7×7 diagonal matrix of the form $< d,d,d,d,1,1,-1 >$ (i.e. of $< d,d,d > \oplus\ p_d$). Then a direct check shows that $A_dFA_d^t = Q_d$ as is required. $\square$

Remark: The appearance of the matrix A_d is described by the following. Let $\Omega = \{w + xi + yj + zij\ :\ w,x,y,z \in \mathbf{Z}\}$ denote the ring of integral Hamiltonian quaternions. If $\alpha = w+xi+yj+zij \in \Omega$, then the norm of α is $d = w^2 + x^2 + y^2 + z^2$. By considering the representation of Ω into $M(4,\mathbf{Q})$ determined by the right action of Ω on itself, α is mapped to the 4×4 block of the matrix A_d, and $\overline{\alpha} = w - xi - yj - zij$, to the transpose of this block.

Proof of Theorem 1.1. By Theorem 3.1 and Lemma 3.4 we deduce that $\mathrm{SO}_0(f;\mathbf{Z})$ is H-subgroup separable on all its geometrically finite subgroups H. By Lemma 4.4 the groups G_d are subgroups of $\mathrm{SO}_0(f;\mathbf{Z})$, and so by Lemma 4.3 they, and all their geometrically finite subgroups (as groups acting on $\mathbf{H}^3$) are geometrically finite subgroups of $\mathrm{SO}_0(f;\mathbf{Z})$ (acting on $\mathbf{H}^6$). Hence Lemma 4.1 shows that G_d is H-subgroup separable for all geometrically finite subgroups H of G_d. Lemma 4.2 allows us to promote this subgroup separability to the groups $\mathrm{PSL}(2,O_d)$. This proves Theorem 1.1. $\square$

4.3 The cocompact case.

We can extend the techniques used in the proof of Theorem 1.1 to cocompact groups. The crucial lemma is:

Lemma 4.6 *Let f be the quadratic form $< 1,1,1,1,-1 >$, and q the quadratic form $< 1,1,1,1,-\phi >$ (as in Lemma 3.3). Then,*

1. $\mathrm{SO}_0(f;\mathbf{Z})$ *contains infinitely many commensurability classes of cocompact arithmetic Kleinian groups. Furthermore these can be chosen to be incommensurable with any group generated by reflections in dimension* 3.

2. $\mathrm{SO}_0(q;\mathcal{O})$ *contains infinitely many commensurability classes of cocompact arithmetic Kleinian groups. Furthermore these can be chosen to be incommensurable with any group generated by reflections in dimension* 3.

The lemma will be proved in §6, assuming it we establish Theorem 1.5.

Proof of Theorem 1.5. The proof is identical to that of the implication of 1.1 from Lemma 4.4. In this case we use Theorem 3.1, Lemma 3.2 and Lemma 3.3 to get the appropriate all right reflection group, and commensurability with the special orthogonal groups in question. $\square$

5 Preliminaries for Lemma 4.6

We use this section to record facts about equivalence of quadratic forms that we will need, (see [14]).

Let K denote either a number field or a completion of a number field, and q a non-singular quadratic form defined over K with associated symmetric matrix Q. The *determinant* of q is the element $d(q) = \det(Q)\dot{K}^2$, where $\dot{K}$ are the invertible elements in K. It is not hard to see that $d(q)$ is an invariant of the equivalence class of q.

The *Hasse invariant* (see [14], p. 122) of a non-singular diagonal form $< a_1, a_2, \dots, a_n >$ with coefficients in K is an element in the Brauer group $B(K)$, namely

$$s(q) = \prod_{i<j} \left(\frac{a_i, a_j}{K} \right)$$

where $\left(\frac{a_i, a_j}{K} \right)$ describes a quaternion algebra over K, and the multiplication is that in $B(K)$, see [14], Chapter 4.

Every non-singular form over K is equivalent over K to a diagonal one, and the definition of the Hasse invariant is extended to non-diagonal forms by simply defining it to be the Hasse invariant of a diagonalization (that this is well-defined follows from [14], p. 122) The following theorem is important to us. It is called the "Weak Hasse-Minkowski Principle" in [14], p. 168. We state it in the case when K is a number field.

Theorem 5.1 *Let q_1 and q_2 be non-singular quadratic forms of the same dimension, defined over K with the property that if σ is a real embedding of K the forms q_1^σ and q_2^σ have the same signature over $\mathbf{R}$. Then q_1 is equivalent to q_2 over K if and only if $d(q_1) = d(q_2)$ and $s(q_1) = s(q_2)$ over all non-archimedean completions of K.* $\square$

Note that if $d(q_1) = d(q_2)$ (resp. $s(q_1) = s(q_2)$) then the same holds locally.

6 Proof of Lemma 4.6(1)

In this section we give the proof of the first part of Lemma 4.6. The method of proof of the third part is the same, however some additional algebraic complexities are involved since we are working over the field $\mathbf{Q}(\sqrt{5})$. This is dealt with in the next section.

6.1

Assume that j is a diagonal quaternary quadratic form with integer coefficients of signature $(3,1)$; so that j is equivalent over $\mathbf{R}$ to the form $<1,1,1,-1>$. Let $a \in \mathbf{Z}$ be a square-free positive integer and consider the five dimensional form $j_a = <a> \oplus j$, where $\oplus$ denotes orthogonal sum. As in §4, if we consider the 5-dimensional $\mathbf{Q}$-vector space V equipped with the form j_a there is a natural 4-dimensional subspace V_0 for which the restriction of the form is j. As before it easily follows that,

Lemma 6.1 *In the notation above, the group* $\mathrm{SO}(j;\mathbf{Z})$ *is a subgroup of* $\mathrm{SO}(j_a;\mathbf{Z})$. $\square$

We begin the proof of the first claim in Lemma 4.6. Let p_d denote the quadratic form $<1,1,1,-d>$. This has signature $(3,1)$, and as discussed in §2 gives arithmetic Kleinian groups. We have the following classical result from number theory, see [14], pp. 173–174.

Theorem 6.2 *Let d be a positive integer. Then d is the sum of three squares if and only if d is not of the form $4^t(8k-1)$.* $\square$

Choose a square-free positive integer $d = -1 \bmod 8$, and let p_d the form $<1,1,1,-d>$. Since a non-trivial rational solution $p_d(x) = 0$ can be easily promoted to an integral solution, Theorem 6.2, shows this form does not represent 0 non-trivially over $\mathbf{Q}$. Hence the arithmetic Kleinian groups $\mathrm{SO}_0(p_d;\mathbf{Z})$ are cocompact. By Theorem 2.3, to get the Kleinian groups to be incommensurable, we simply insist further that d is a prime. By Dirichlet's Theorem there are infinitely many such primes. With these remarks, it follows from Theorem 2.3 that the groups $\mathrm{SO}(p_d;\mathbf{Z})$ are all incommensurable. The first part of Lemma 4.6 will follow from.

Lemma 6.3 *Let $q_d = <d> \oplus p_d$. Then q_d is equivalent over $\mathbf{Q}$ to f.*

Proof. The two forms are 5-dimensional, and it is easy to see that the forms have signature $(4,1)$ over $\mathbf{R}$. Further since the determinants are $-1\dot{\mathbf{Q}}^2$, they will have the same local determinants. We shall show that the forms have the same Hasse invariants over $\mathbf{Q}$ from which it follows they have the same local Hasse invariants. Theorem 5.1 completes the proof.

Consider the form f first of all. It is easy to see that all the contributing terms to the product are either $\left(\frac{1,1}{\mathbf{Q}}\right)$ or $\left(\frac{-1,1}{\mathbf{Q}}\right)$. Both of these are isomorphic to the quaternion algebra of 2×2 matrices over $\mathbf{Q}$, see [14] Chapter 3. These represent the trivial element in the Brauer group of $\mathbf{Q}$, and so $s(f)$ is trivial.

For q_d the contributing terms are

$$\left(\frac{1,1}{\mathbf{Q}}\right),\ \left(\frac{1,d}{\mathbf{Q}}\right),\ \left(\frac{1,-d}{\mathbf{Q}}\right),\left(\frac{d,-d}{\mathbf{Q}}\right).$$

From [14] Chapter 3, in particular p. 60, it follows that all these quaternion algebras are again isomorphic to the quaternion algebra of 2×2 matrices over $\mathbf{Q}$, and so as above $s(q_d)$ is trivial. This completes the proof. $\square$

Remark: The proof of this Lemma can also be done directly as in the case of Lemma 4.4. We include this version, as it may be useful as a guide to the proof of Lemma 7.6.

To complete the proof of the first claim in Lemma 4.6 we proceed as follows—entirely analogous to the argument in the proof of 1.1. By Lemma 6.1, $\mathrm{SO}(p_d;\mathbf{Z})$ is a subgroup of $\mathrm{SO}(q_d;\mathbf{Z})$. By Lemma 6.3 and Lemma 2.2 we can conjugate to obtain a group $G_d < SO(f;\mathbf{Z})$ which is conjugate to a

subgroup of finite index in $\mathrm{SO}(p_d; \mathbf{Z})$. Finally, the groups constructed are not commensurable with groups generated by reflections for d large enough. This follows from work of Nikulin, [19].

Briefly it is shown in [19] that if we fix the field of definition for a reflection group (in this case $\mathbf{Q}$) there are only finitely many commensurability classes of arithmetic Kleinian groups commensurable with a group generated by reflections. This completes the proof. $\square$

Remarks.
1. The argument given in Lemma 4.6 also applies in the case of the Bianchi groups.
2. An example of an explicit "well known" co-compact group that is covered by our techniques arises in the choice of $d = 7$. It follows from [18] that the arithmetic Kleinian group Γ arising as an index 2 subgroup in the group generated by reflections in the faces of the tetrahedron $T_6[2,3,4;2,3,4]$ is commensurable with the group $\mathrm{SO}(p_7; \mathbf{Z})$.

Also commensurable with $\mathrm{SO}(p_7; \mathbf{Z})$ is the fundamental group of a certain non-Haken hyperbolic 3-manifold obtained by filling on a once-punctured torus bundle. Briefly, let M denote the once-punctured torus bundle whose monodromy is R^2L^2 (in terms of the usual RL-factorization, see [11] for instance). As is well-known, M contains no closed embedded essential surfaces. Thus a Haken manifold can be created by Dehn filling on M, only by filling along a boundary slope of M. Fixing a framing for the boundary torus, the manifold M_0 obtained by 4/1-Dehn filling on M is hyperbolic, of volume approximately $2.666744783449061\ldots$, and non-Haken (the boundary slopes can be deduced from [11]). A calculation with Snap (see [9] for a discussion of this program) shows M_0 is arithmetic with the same invariant data as the group Γ above. Hence $\pi_1(M_0)$ is separable on geometrically finite subgroups.

7 Proof of Lemma 4.6(2)

To handle the second part, we proceed in a similar way to §6.

Thus, assume that j is a diagonal quaternary quadratic form with coefficients in $\mathcal{O}$, of signature $(3,1)$ at the identity and signature $(4,0)$ on applying τ. Let $a \in \mathcal{O}$ be postive at both the identity embedding and τ. Consider the five dimensional form $j_a =< a > \oplus j$, where $\oplus$ denotes orthogonal sum. As above, we mean 5-dimensional $\mathbf{Q}(\sqrt{5})$-vector space V equipped with the form j_a there is a natural 4-dimensional subspace V_0 for which the restriction of the form is j. We have the following consequence of the discussion in §2.2.

Lemma 7.1 *Let $a \in \mathcal{O}$ have the property that it is square-free (as an element of $\mathcal{O}$), $a < 0$ but $\tau(a) > 0$. Define the $n+1$-dimensional form*

$$f_{n,a} =< 1, 1, \ldots, 1, a > .$$

Then $\mathrm{SO}_0(f_{n,a}; \mathcal{O})$ defines a cocompact arithmetic subgroup of $Isom_+(\mathbf{H}^n)$. $\square$

With this, we deduce as in §6.1,

Lemma 7.2 *In the notation above, the group $\mathrm{SO}(j; \mathcal{O})$ is a cocompact subgroup of $\mathrm{SO}(j_a; \mathcal{O})$.* $\square$

7.1

We need to recall some basic number theory in $\mathcal{O}$. $\mathcal{O}$ is a principal ideal domain, so every ideal of $\mathcal{O}$ has the form $< t >$ for some $t \in \mathcal{O}$.

Lemma 7.3 *Let $I \subset \mathcal{O}$ be a non-trivial ideal. Then I can be generated by an element t where $t < 0$ and $\tau(t) > 0$.*

Proof. Assume $I =< x >$. The argument is really a consequence of the existence of units of all possible signatures in $\mathcal{O}$. For example if x is positive at both the identity and τ, then $t = x(\frac{1+\sqrt{5}}{2})$ satisfies the requirements. As $\frac{1+\sqrt{5}}{2}$ is a unit it does not change the ideal. $\square$

Define $\mathbf{P}$ to be the set of primes in $\mathbf{Q}(\sqrt{5})$ lying over the set of rational primes $\mathcal{Q}$ satisfying:

1. if $q \in \mathcal{Q}$ then q is unramified in $\mathbf{Q}(\sqrt{\frac{1+\sqrt{5}}{2}})$,
2. the splitting type of q in $\mathbf{Q}(\sqrt{\frac{1+\sqrt{5}}{2}})$ contains a prime of degree 1, ie. there is a $\mathbf{Q}(\sqrt{\frac{1+\sqrt{5}}{2}})$-prime Q with $Q|q$ and the norm of Q is q.

Note by the Tchebotarev density theorem (see [15] for example), the set $\mathbf{P}$ is infinite, since the set of rational primes that split completely in $\mathbf{Q}(\sqrt{\frac{1+\sqrt{5}}{2}})$ is infinite. Fix $\phi = \frac{1+\sqrt{5}}{2}$ in what follows.

Lemma 7.4 *Let $\mathcal{P} \in \mathbf{P}$. Then under the canonical reduction map $\mathcal{O} \to \mathcal{O}/\mathcal{P}$, ϕ is a square.*

Proof. Let $\mathcal{R}$ denote the ring of integers in $\mathbf{Q}(\sqrt{\frac{1+\sqrt{5}}{2}})$. If $p \in \mathcal{Q}$, then since $\mathbf{Q}(\sqrt{5})$ is galois, p splits completely in $\mathbf{Q}(\sqrt{5})$. By assumption there is a degree one prime $P \subset \mathcal{R}$ with $P|p\mathcal{R}$. Let $\mathcal{P} \in \mathbf{P}$ be a $\mathbf{Q}(\sqrt{5})$-prime divisible by P. Then the ramification theory of primes in extensions (see [15]), together with the above properties give,

$$\mathcal{R}/P \cong \mathcal{O}/\mathcal{P} \cong \mathbf{F}_p.$$

Hence by definition, ϕ has a square-root upon reduction mod $\mathcal{P}$ as required. $\square$

Define the following collection of quadratic forms over $\mathbf{Q}(\sqrt{5})$. Let $\mathcal{P} \in \mathbf{P}$ be generated by an element π satisfying the conclusion of Lemma 7.3, that is

$$\pi < 0 \text{ and } \tau(\pi) > 0.$$

Define $p_\pi =< 1, 1, 1, \pi >$. Thus, by Lemma 7.1, $\mathrm{SO}(p_\pi; \mathcal{O})$ determines a cocompact arithmetic Kleinian group. Furthermore this infinite collection of groups are all mutually incommensurable. This follows [3] or [18] (this is the generalization of Theorem 2.3 to $\mathbf{Q}(\sqrt{5})$).

Define the 5-dimensional form $q_\pi =< -\pi\phi > \oplus p_\pi$. Since π (resp. $\tau(\pi)$) is negative (resp. positive) and ϕ (resp. $\tau(\phi)$) is positive (resp. negative), both $-\pi\phi$ and $-\tau(\pi)\tau(\phi)$ are positive. Thus q_π has signature $(4, 1)$ at the identity and $(5, 0)$ at τ. Therefore, by Lemma 7.2, $\mathrm{SO}(q_\pi; \mathcal{O})$ determines a cocompact arithmetic subgroup of $\mathrm{Isom}_+(\mathbf{H}^4)$, which contains $\mathrm{SO}(p_\pi; \mathcal{O})$.

In what follows we quote freely from the theory of quaternion algebras, see [14] Chapter 3 and [25].

We will need the following.

Lemma 7.5 *Let $k = \mathbf{Q}(\sqrt{5})$. The quaternion algebra $B = \left(\frac{\phi, \pi}{k}\right)$ is isomorphic to $M(2, k)$.*

Proof. Note that by choice of π, B is unramified at both the identity embedding and τ. Now Lemma 7.4 implies ϕ is a square upon reduction mod π. It follows that the norm form of B, namely $< 1, -\phi, -\pi, \phi\pi >$ is isotropic over $k_{<\pi>}$, since it is isotropic over the residue class field (see [14] Chapter 6). The only primes that can ramify B are $< \pi >$ and the unique prime above 2 in k (cf. [25] or [14] Chapter 6). Furthermore since the cardinality of the ramification set is even ([25]), we deduce that B is unramified at the prime above 2. Hence B is unramified everywhere locally, and so is isomorphic to $M(2, k)$. $\square$

Lemma 7.6 *In the notation above, q_π is equivalent over $\mathbf{Q}(\sqrt{5})$ to q.*

Proof. Let $k = \mathbf{Q}(\sqrt{5})$. The two forms are 5-dimensional, and as noted both forms have signature $(4,1)$ at the identity embedding of k, and signature $(5,0)$ at τ. Further since the determinants are $-\phi\dot{k}^2$, they will have the same local determinants. We shall show that the forms have the same Hasse invariants over k from which it follows they have the same local Hasse invariants. Theorem 5.1 completes the proof.

Consider the form q first of all. It is easy to see that all the contributing terms to the product are either $\left(\frac{1,1}{k}\right)$ or $\left(\frac{1,-\phi}{k}\right)$. Both of these are isomorphic to the quaternion algebra of 2×2 matrices over k, see [14] Chapter 3 (in particular p. 60). These represent the trivial element in the Brauer group of k, and so $s(f)$ is trivial.

For q_π the contributing terms are

$$\left(\frac{1,1}{k}\right),\ \left(\frac{1,-\phi\pi}{k}\right),\ \left(\frac{1,\pi}{k}\right),\left(\frac{\pi,-\phi\pi}{k}\right).$$

As above, it follows from [14] Chapter 3, p. 60, that all but the last Hilbert symbol represent quaternion algebras isomorphic to the quaternion algebra of 2×2 matrices over k.

Standard Hilbert symbol manipulations ([14] Chapter 3) imply this last algebra is isomorphic to one with Hilbert Symbol $\left(\frac{\phi,\pi}{k}\right)$. But Lemma 7.5 implies this quaternion algebra is the matrix algebra again. Hence $s(q_\pi)$ is also trivial. This completes the proof. □

To complete the proof of Lemma 4.6(3) we proceed as follows. As noted all groups are cocompact, and Lemma 7.2 implies that $\mathrm{SO}(p_\pi;\mathcal{O})$ is a subgroup of $\mathrm{SO}(q_\pi;\mathcal{O})$. By Lemma 6.3 and Lemma 2.2 we can conjugate to obtain a group $G_\pi < \mathrm{SO}(f;\mathcal{O})$ which is conjugate to a subgroup of finite index in $\mathrm{SO}(p_\pi;\mathcal{O})$, The final part is to deduce that infinitely many of these are not commensurable with groups generated by reflections, and again this follows from [19], with the ground field in this case being $\mathbf{Q}(\sqrt{5})$. □

Corollary 1.6 is deduced by choosing $\pi = (3-2\sqrt{5})$ in the construction above. Note $<\pi>$ is a prime in $\mathbf{Q}(\sqrt{5})$ of norm 11, and as is easily checked 11 is unramified in $\mathbf{Q}(\sqrt{\phi})$ and also satisfies the second condition in the definition of $\mathbf{P}$ in §4.2. It follows from [18] that the arithmetic Kleinian group Γ_0 is commensurable with $\mathrm{SO}(p_\pi;\mathcal{O})$. □

Corollary 1.7 is deduced in a similar manner. The Seifert-Weber dodecahedral space is constructed from 120 copies of the tetrahedron $T_4[2,2,5;2,3,5]$. If we let Γ denote the group generated by reflections in faces of this tetrahedron, the results of [18] imply Γ is commensurable (up to conjugacy) with the group $\mathrm{SO}(f;\mathcal{O})$ where f is the form $<1,1,1,-1-2\sqrt{5}>$. Now $<-1-2\sqrt{5}>$ generates a prime ideal of norm 19, and, as is easily checked, lies in $\mathbf{P}$. The result now follows. □

8 Application.

Theorem 1.1 allows to address the following question of Lubotzky raised in [17]. We require some terminology. The profinite topology on a group G is defined by proclaiming all finite index subgroups of G to be a basis of open neighbourhoods of the identity. Let $\hat{G}$ denote the profinite completion of G.

An obvious subgroup of $\mathrm{PSL}(2, O_d)$ is $\mathrm{PSL}(2, \mathbf{Z})$, and in the context of the Congruence Kernel, Lubotzky [17] asked the following question:

Question: Is the induced map:

$$\eta \;:\; \widehat{\mathrm{PSL}}(2, \mathbf{Z}) \to \widehat{\mathrm{PSL}}(2, O_d)$$

injective?

Since open subgroups are closed in the profinite topology, it is not hard to see that G is H-subgroup separable if and only if H is closed in the profinite topology on G. Thus an equivalent formulation of Theorem 1.1 is that if H is a geometrically finite subgroup of $\mathrm{PSL}(2, O_d)$, then on passing to the profinite completion $\widehat{\mathrm{PSL}}(2, O_d)$, we require that the only points of $\mathrm{PSL}(2, O_d)$ in the closure of H (in $\widehat{\mathrm{PSL}}(2, O_d)$) are points of H. This formulation can be can be used to give an affirmative answer.

Theorem 8.1 *The map η_d is injective for all* $\mathrm{PSL}(2, O_d)$

Proof. The group $\mathrm{PSL}(2, O_d)$ is residually finite, so that it (and hence $\mathrm{PSL}(2, \mathbf{Z})$) embeds into the profinite completion $\widehat{\mathrm{PSL}}(2, O_d)$. Thus we may take the closure of $\mathrm{PSL}(2, \mathbf{Z}) \subset \widehat{\mathrm{PSL}}(2, O_d)$ to form a completion, denoted $\overline{\mathrm{PSL}}(2, \mathbf{Z})$. This completion is obviously embedded in $\widehat{\mathrm{PSL}}(2, O_d)$, but it is potentially coarser than the genuine profinite completion $\widehat{\mathrm{PSL}}(2, \mathbf{Z})$.

However any finitely generated subgroup H (in particular any subgroup of finite index) in $\mathrm{PSL}(2, \mathbf{Z})$ is geometrically finite and thus by Theorem 1.1 separable in $\mathrm{PSL}(2, O_d)$ and it follows easily that one can find a subgroup H^* of finite index in $\mathrm{PSL}(2, O_d)$ with the property that $H^* \cap \mathrm{PSL}(2, \mathbf{Z}) = H$ so that in fact the map $\widehat{\mathrm{PSL}}(2, \mathbf{Z}) \to \overline{\mathrm{PSL}}(2, \mathbf{Z})$ is a homeomorphism, proving the theorem. □

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